Radiation Losses of Tapered Dielectric Slab Waveguides

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In this paper we calculate radiation losses of a single mode dielectric slab waveguide for TE and TM modes. The theory is based on the determination of the radiation losses of one abrupt step. We obtain the losses of arbitrarily deformed waveguides by regarding the arbitrary deformations as a succession of infinitely many infinitesimal steps. This method yields the same results as a very different method presented earlier. It allows us to calculate the losses of TM modes that were hard to obtain by the earlier method.

The radiation losses of single mode slab waveguides with abrupt steps of a 2:1 ratio are surprisingly low and can be kept below 1 percent by dimensioning the guide properly. The loss advantage of linear tapers becomes noticeable only when the tapers are very long. An optimized taper changes more rapidly in its wider portion and becomes more gradual in its narrow part.

I. INTRODUCTION

The study of radiation losses of dielectric waveguides, which has been described in three earlier papers,[1-3] has been extended to cover abrupt steps in a single mode waveguide as well as continuous tapers. The mathematical theory of radiation losses caused by a step in the waveguide is used to compute the losses caused by tapers by regarding the taper as a succession of infinitely many infinitesimal steps. This method can also be used to rederive the equations for a dielectric slab waveguide with small wall distortions presented earlier.[1] Both the earlier method and the derivation based on small steps lead to identical results. The perturbation theory used in Ref. 1 was not very well suited for calculating the losses of TM modes. The step method is equally applicable to TM and TE modes and allows us to derive for TM modes

the corresponding expressions which for TE modes were presented in Ref. 1.

The radiation losses of steps and tapers are surprisingly small. A step which changes the thickness of a dominant mode slab waveguide to one half of its original value causes a loss of only about 1 percent for TE modes and about 2 percent for TM modes if operated at favorable frequencies. The losses of tapers are even smaller and can be made as small as desired for sufficiently long tapers.

Comparison of the radiation losses of slab waveguides with round and rectangular waveguides (to be published) shows that the slab waveguide losses are exceptionally low. The losses caused by steps in circular waveguides are higher by an order of magnitude.

II. THE MODES OF THE SLAB WAVEGUIDE

We state briefly the TE and TM modes of the dielectric slab waveguide. For simplicity we assume that all the fields are independent of one spatial coordinate so that we can write symbolically

$$\frac{\partial}{\partial y} = 0. \tag{1}$$

Incidentally, it is only because we limit the discussion to cases where equation (1) applies that it is possible to speak of transverse electric (TE) and transverse magnetic (TM) modes. In the general case the modes are hybrids and possess longitudinal E as well as H components. The modes of the dielectric slab waveguide consist of a finite set of guided modes and a continuum of radiation modes. The slab geometry is shown in Fig. 1.

2.1 *TE Modes*

The field components E_x, E_z and H_y vanish. The remaining components of the magnetic field can be obtained from E_y

$$H_x = -\frac{i}{\omega\mu}\frac{\partial E_y}{\partial z} = -\frac{\beta}{\omega\mu}E_y \tag{2}$$

$$H_z = \frac{i}{\omega\mu}\frac{\partial E_y}{\partial x}. \tag{3}$$

The dependence of the field component on the length coordinate z and on the time t is given by

$$e^{i(\omega t - \beta z)}. \tag{4}$$

This factor will be omitted from the following equations.

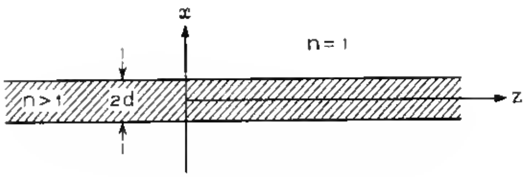

Fig. 1 — Dielectric slab waveguide.

2.1.1 *Even Guided Modes*

$$\left.\begin{aligned} E_y &= A_e \cos \kappa x & |x| \leqq d \\ E_y &= A_e e^{\gamma d} \cos \kappa d e^{-\gamma |x|} & |x| \geqq d \end{aligned}\right\}. \tag{5}$$

The coefficient A_e is related to the power P carried by the mode by the following equation

$$A_e = \left\{\frac{2\omega\mu_0 P}{\beta_0\, d + \dfrac{\beta_0}{\gamma}}\right\}^{\frac{1}{2}}. \tag{6}$$

The relation between κ, γ and β_0 is given by

$$\kappa = [(nk)^2 - \beta_0^2]^{\frac{1}{2}}, \tag{7a}$$

$$\gamma = [\beta_0^2 - k^2]^{\frac{1}{2}}, \tag{7b}$$

$$k = \omega(\epsilon_0\mu_0)^{\frac{1}{2}}. \tag{8}$$

n is the index of refraction of the dielectric slab. The index of the surrounding medium is taken to be $n = 1$. The eigenvalue equation for the determination of β_0 is

$$\tan \kappa d = \frac{\gamma}{\kappa}. \tag{9}$$

A few numerical values for β_0 are shown in Table I. The TE modes are power orthogonal. With the power flow P in z-direction (per unit length of y) we have

$$P\,\delta_{nm} = \frac{\beta_n}{\omega\mu}\int_0^\infty E_{yn}E_{ym}^*\, dx. \tag{10}$$

2.1.2 *Even Radiation Modes*

$$\left.\begin{aligned} E_y &= B_e \cos \sigma x & |x| \leqq d \\ E_y &= C_e e^{i\rho|x|} + C_e^* e^{-i\rho|x|} & |x| \geqq d \end{aligned}\right\}. \tag{11}$$

TABLE I—SOME NUMERICAL VALUES OF β_0

kd	n	TE Mode $\beta_0 d$	TM Mode $\beta_0 d$
2.5		2.50271	2.50263
5.0		5.01550	5.01519
10.0	1.01	10.06061	10.06016
20.0		20.16711	20.16680
0.25		0.25781	0.25207
0.5		0.54916	0.51677
1.0	1.432	1.21972	1.12809
1.5		1.93825	1.84210
2.0		2.66839	2.58934
3.0		4.13075	4.08131

Propagation constants of TE and TM modes

(The asterisk indicates the complex conjugate value) with

$$\sigma = [(nk)^2 - \beta^2]^{\frac{1}{2}}, \tag{12}$$

$$\rho = [k^2 - \beta^2]^{\frac{1}{2}}, \tag{13}$$

$$C_e = \tfrac{1}{2} B_e \exp(-i\rho d)\left(\cos \sigma d + i \frac{\sigma}{\rho} \sin \sigma d\right), \tag{14}$$

$$B_e = \left\{\frac{2\rho^2 \omega\mu P}{\pi\beta(\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d)}\right\}^{\frac{1}{2}}. \tag{15}$$

The power orthogonality of the radiation modes can be expressed by the equation

$$P\delta(\rho - \rho') = \frac{\beta}{\omega\mu} \int_0^\infty E_y(x, \rho) E_y^*(x, \rho')\, dx. \tag{16}$$

P is the power flowing per unit length (in y-direction) in the z-direction.

The odd TE modes have been listed in Ref. 1 (together with the even TE modes). Since we are limiting the discussion of TE modes to symmetrical tapers excited by an even mode we will not need the odd TE modes in this paper.

2.2 *TM Modes*

With the restriction imposed by equation (1) the only nonvanishing components of the TM modes are H_y,

$$E_x = \frac{i}{\omega\epsilon} \frac{\partial H_y}{\partial z}, \tag{17}$$

$$E_z = -\frac{i}{\omega\epsilon} \frac{\partial H_y}{\partial x}. \tag{18}$$

We have no occasion to use the odd guided TM modes, therefore only the even guided modes will be listed.

2.2.1 *Even Guided Modes*

$$\left.\begin{aligned} H_y &= A_e \cos \kappa x & &\text{for} \quad |x| \leqq d \\ H_y &= A_e e^{\gamma d} \cos \kappa d e^{-\gamma |x|} & &\text{for} \quad |x| \geqq d \end{aligned}\right\}. \tag{19}$$

The amplitude constant is related to tbe power P carried by the mode

$$A_e = \left\{ \frac{\gamma}{\beta_0} \frac{2\omega\epsilon P}{\dfrac{n^2 k^2}{\beta_0^2 + n^2\gamma^2} + \gamma d} \right\}^{\frac{1}{2}}. \tag{20}$$

The constants κ and β are related to β_0 by equations (7a) and (7b). The eigenvalue β_0 of the even guided TM modes is obtained as a solution of the eigenvalue equation

$$\tan \kappa d = n^2 \frac{\gamma}{\kappa}. \tag{21}$$

A few numerical values for β_0 are shown in Table I. The power orthogonality of the guided TM modes can be expressed by

$$P\,\delta_{nm} = \frac{\beta_n}{\omega} \int_0^\infty \frac{1}{\epsilon} H_{yn} H_{ym}^* \, dx = \frac{\omega}{\beta_n} \int_0^\infty \epsilon E_{xn} E_{xm}^* \, dx. \tag{22}$$

2.2.2 *Even Radiation Modes*

$$\left.\begin{aligned} H_y &= B_e \cos \sigma x & & |x| \leqq d \\ H_y &= C_e e^{i\rho|x|} + C_e^* e^{-i\rho|x|} & & |x| \geqq d \end{aligned}\right\} \tag{23}$$

with ρ and γ given by equations (12) and (13) and with

$$C_e = \tfrac{1}{2} B_e \left(\cos \sigma d + \frac{i}{n^2} \frac{\sigma}{\rho} \sin \sigma d \right) e^{-i\rho d}. \tag{24}$$

The amplitude B_e is given by

$$B_e = \rho \left\{ \frac{2\omega\epsilon P}{\pi\beta \left(n^2 \rho^2 \cos^2 \sigma d + \dfrac{\sigma^2}{n^2} \sin^2 \sigma d \right)} \right\}^{\frac{1}{2}}. \tag{25}$$

2.2.3 *Odd Radiation Modes*

$$\left.\begin{aligned} H_y &= B_0 \sin \sigma x & &\text{for} \quad |x| \leqq d \\ H_y &= \frac{x}{|x|} \{ C_0 e^{i\rho|x|} + C_0^* e^{-i\rho|x|} \} & &\text{for} \quad |x| \geqq d \end{aligned}\right\} \tag{26}$$

with

$$C_0 = \tfrac{1}{2}B_0 e^{-i\rho d}\left(\sin \sigma d - \frac{i}{n^2}\frac{\sigma}{\rho}\cos \sigma d\right) \tag{27}$$

and

$$B_0 = \rho\left\{\frac{2\omega\epsilon P}{\pi\beta\left(n^2\rho^2 \sin^2 \sigma d + \frac{\sigma^2}{n^2}\cos^2 \sigma d\right)}\right\}^{\frac{1}{2}}. \tag{28}$$

The power orthogonality of the radiation modes is expressed as

$$P\delta_{e0}\,\delta(\rho - \rho') = \frac{\beta}{2\omega}\int_{-\infty}^{\infty}\frac{1}{\epsilon}H_{ye}(x, \rho)H_{y0}^*(x, \rho')\,dx. \tag{29}$$

All the modes are orthogonal among each other. The amount of power P carried by each mode is normalized to the same value. The actual power carried by the field is determined by the expansion coefficients.

III. TE MODE RADIATION LOSS

Prior to discussing the radiation losses of a waveguide taper we calculate the losses of an abrupt step in the dielectric slab waveguide. We limit our investigation to the case that only the lowest order guided mode of each type exists. These modes do not experience a cut-off and can exist on waveguides with vanishingly small thickness. The steps are considered to be sufficiently small to keep the guide dimensions below the point where a second guided TE or TM mode becomes possible.

The geometry of the step is shown in Fig. 2. The loss problem is solved by assuming that one guided (TE or TM) mode is incident on the step. The discontinuity in the waveguide causes a reflected mode as well as forward and backward traveling radiation modes to occur. The unknown amplitudes of these modes are determined by requiring

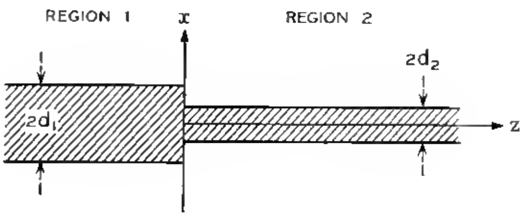

Fig. 2 — Abrupt step in a dielectric slab waveguide.

that the transverse field components are continuous at the step. For TE modes we get the following equations:

$$\begin{aligned} E_y^{(i)} + a_r E_y^{(r)} + \int_0^\infty q_r(\rho) E_y^{(r)}(\rho)\, d\rho \\ = c_t E_y^{(t)} + \int_0^\infty q_t(\rho) E_y^{(t)}(\rho)\, d\rho, \end{aligned} \tag{30}$$

$$\begin{aligned} H_x^{(i)} + a_r H_x^{(r)} + \int_0^\infty q_r(\rho) H_x^{(r)}(\rho)\, d\rho \\ = c_t H_x^{(t)} + \int_0^\infty q_t(\rho) H_x^{(t)}(\rho)\, d\rho. \end{aligned} \tag{31}$$

The superscripts i, r and t indicate incident, reflected and transmitted waves. The field components whose ρ dependence is explicitly shown are radiation modes, the other field components belong to guided modes.

There are two ways to compute the radiation losses. We can calculate the coefficients c_t and a_r of the transmitted and reflected guided mode and calculate the radiated power loss from

$$\frac{\Delta P}{P} = 1 - |c_t|^2 - |a_r|^2 \tag{32}$$

or we can calculate the coefficients q_t and q_r and obtain the radiation losses from

$$\frac{\Delta P}{P} = \int_{-k}^{0} |q_r|^2 \frac{|\beta|}{\rho}\, d\beta + \int_0^k |q_t|^2 \frac{\beta}{\rho}\, d\beta. \tag{33}$$

Both methods should, of course, lead to the same result.

It is impossible to obtain exact solutions of equations (30) and (31); a comparison of both methods (32) and (33) allows an estimate of the validity of the approximations that are used to solve these equations.

We obtain approximate solutions by the following argument. Since all modes of the same waveguide section are orthogonal we can use the orthogonality of the modes to isolate c_t on the right hand side of equations (30) and (31). We get for TE modes from (30)

$$c_t = \frac{\beta_2}{\omega\mu P}(1 + a_r) \int_0^\infty E_y^{(i)} E_y^{(t)*}\, dx \tag{34a}$$

and from equation (31)

$$c_t = \frac{\beta_1}{\omega\mu P}(1 - a_r) \int_0^\infty E_y^{(i)} E_y^{(t)*}\, dx. \tag{34b}$$

The coefficient q_r was neglected. For large steps the radiation is scattered predominantly in forward direction so that q_r is indeed small. If the step height is small the fields $E_y^{(t)}$ and $E_y^{(r)}(\rho)$ become more nearly orthogonal so that q_r again does not contribute very much to equations (34a) and (34b). The propagation constant β_2 belongs to the guided mode on the waveguide to the right of the step while β_1 belongs to the guided mode to the left of the step. Because of the different waveguide size these propagation constants are not the same.

Equations (34a) and (34b) allow the determination of c_t and a_r

$$c_t = \frac{2\beta_1\beta_2}{\beta_1 + \beta_2}\frac{1}{\omega\mu P}\int_0^\infty E_y^{(i)}E_y^{(t)*}\,dx, \tag{35}$$

$$a_r = \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2}. \tag{36}$$

The integral can be evaluated with the help of equation (5) so that we obtain

$$c_t = \frac{4(n^2 - 1)\beta_1\beta_2 k^2 \cos\kappa_2 d_2}{\left[\left(\beta_1 d_1 + \dfrac{\beta_1}{\gamma_1}\right)\left(\beta_2 d_2 + \dfrac{\beta_2}{\gamma_2}\right)\right]^{\frac{1}{2}}(\beta_1 + \beta_2)^2(\beta_1 - \beta_2)(\kappa_1^2 + \gamma_2^2)}$$
$$\cdot[\gamma_2 \cos\kappa_1 d_2 - \kappa_1 \sin\kappa_1 d_2 + (\gamma_1 - \gamma_2)\cos\kappa_1 d_1 e^{-\gamma_2(d_1 - d_2)}]. \tag{37}$$

The determination of q_r and q_t is not quite as simple. The functions $E_y^{(r)}(\rho)$ and $E_y^{(t)}(\rho)$ belong to different waveguides and are not orthogonal. For large steps with predominantly forward scattering q_r may again be negligible but this is certainly not true for small steps. We would need different approximations for large and small steps. To avoid this difficulty we consider only small steps and construct large steps and waveguide tapers as a succession of small steps. For infinitesimal steps the modes $E_y^{(r)}$ and $E_y^{(t)}$ are very nearly orthonormal and reflected guided modes can be neglected. Using the orthogonality of the modes we obtain

$$q_t(\rho) = \tfrac{1}{2}(\beta_0 + \beta)I \tag{38}$$

and

$$q_r(\rho) = \tfrac{1}{2}(\beta_0 - \beta)I \tag{39}$$

with

$$I = \frac{1}{\omega\mu P}\int_0^\infty E_y^{(i)}E_y^{(t)*}(\rho)\,dx. \tag{40}$$

The expression I does not depend on the sign of β, we therefore obtain

$q_r(\rho)$ from $q_t(\rho)$ by reversing the sign of the propagation constant β of the radiation mode. We may drop the subscript r and t and obtain after integration

$$q(\rho) = -(n^2 - 1)k^2 \cdot \frac{\rho \cos \kappa d \cos \sigma d \, \Delta d}{(\pi)^{\frac{1}{2}}(\beta_0 - \beta)\left[\, | \beta | \left(\beta_0 \, d + \frac{\beta_0}{\gamma_0}\right)(\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d)\right]^{\frac{1}{2}}}. \quad (41)$$

The difference $\Delta d = d_2 - d_1$ is assumed to be small. Because of the relation between $q_r(\rho)$ and $q_t(\rho)$ we can write equation (33) more simply

$$\frac{\Delta P}{P} = \int_{-k}^{k} | \, q(\rho) \, |^2 \frac{| \beta |}{\rho} \, d\beta. \quad (42)$$

IV. APPLICATION TO TAPERS

Equation (41) can immediately be extended to apply to symmetrical waveguide distortions of arbitrary shape. We assume that the shape of the waveguide wall is described by the function $f(z)$ as shown in Fig. 3. We can then write

$$\Delta d = \frac{df}{dz} \, dz. \quad (43)$$

The amplitude $q(\rho)$ was calculated for a small step at $z = 0$. Locating the step at z the guided wave arrives there with the phase $e^{-i\beta_0 z}$ instead of with phase zero as assumed in equation (41). The radiation mode was also referred to $z = 0$. Referring it to a step at z adds the phase factor $e^{i\beta z}$ to equation (41) because the amplitude B of the radiation mode enters equation (41) with its complex conjugate value. A step at z would be described by an expression like (41) with an additional phase factor

$$e^{-i(\beta_0 - \beta)z}. \quad (44)$$

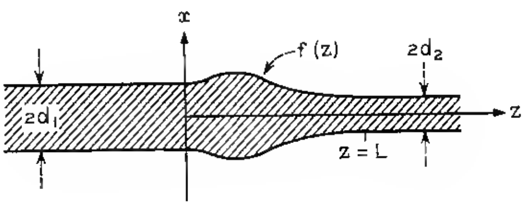

Fig. 3 — A symmetrical wall distortion (symmetrical taper) of a dielectric slab waveguide.

It must be assumed that β_0 (but not β) is a function of z if the guide thickness is changing.

The total radiation loss of a section of waveguide (for example a taper) of length L is given by equation (42) with

$$q(\rho) = -(n^2 - 1)k^2 \cdot \int_0^L \frac{\rho \cos \kappa d \cos \sigma d e^{-i(\beta_0 - \beta)z} \dfrac{df}{dz}}{(\beta_0 - \beta)\left[\pi \mid \beta \mid \left(\beta_0 d + \dfrac{\beta_0}{\gamma_0}\right)(\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d)\right]^{\frac{1}{2}}} dz. \tag{45}$$

Except for the restriction to symmetrical waveguides, equation (45) describes the same problem as treated in Ref. 1. In fact, we can obtain equation (57) of Ref. 1 by a partial integration. The formulation of Ref. 1 applies to the case that the thickness of the waveguide at $z = 0$ and $z = L$ is very nearly the same. The function $f(z)$ deviates so little from the half thickness d of the perfect waveguide that β_0, κ and γ can be assumed to be independent of d. With these assumptions, we obtain as a result of a partial integration

$$q(\rho) = \frac{(n^2 - 1)k^2 \rho \cos \kappa d \cos \sigma d \varphi(\beta)}{i\left[\pi \mid \beta \mid \left(\beta_0 d + \dfrac{\beta_0}{\gamma_0}\right)(\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d)\right]^{\frac{1}{2}}} \tag{46}$$

with

$$\varphi(\beta) = \int_0^L f(z) e^{-i(\beta_0 - \beta)z} \, dz. \tag{47}$$

The agreement with equation (57), Ref. 1, is perfect if we keep in mind that the functions describing the upper and lower side of the waveguide are now identical except for a minus sign and that the function $f(z) - d$ of Ref. 1 is now redefined and replaced by $f(z)$.

The fact that equation (45) is identical to the theory of Ref. 1 proves the validity of our method of continuous steps.

4.1 *TM Mode Radiation Loss*

The radiation losses of the lowest order guided TM mode at a symmetrical step in the dielectric slab waveguide can be calculated from equations (30) and (31) by changing the subscript x to y and y to x.

The c_t coefficient for the lowest order (dominant) even TM mode is

$$c_t = \frac{2I_1I_2}{I_1 + I_2} \tag{48}$$

and the a_r coefficient is

$$a_r = \frac{I_1 - I_2}{I_1 + I_2}, \tag{49}$$

with

$$I_1 = \frac{(n^2 - 1)\beta_1 \cos \kappa_2 d_2}{(\beta_2^2 - \beta_1^2)(\kappa_1^2 + \gamma_2^2)} \left[\frac{4\, \gamma_1 \gamma_2}{\beta_1 \beta_2 \left(\frac{n^2 k^2}{\beta_1^2 + n^2 \gamma_1^2} + \gamma_1 d_1 \right) \left(\frac{n^2 k^2}{\beta_2^2 + n^2 \gamma_2^2} + \gamma_2 d_2 \right)} \right]^{\frac{1}{2}}$$

$$\cdot \{ \kappa_1 k^2 \sin \kappa_1 d_2 - \gamma_2 (\kappa_1^2 + \beta_2^2) \cos \kappa_1 d_2$$
$$+ [\gamma_2 (n^2 k^2 + \beta_2^2 - \beta_1^2) - n^2 \gamma_1 k^2] e^{-\gamma_2 (d_1 - d_2)} \cos \kappa_1 d_1 \}, \tag{50}$$

and

$$I_2 = \frac{(n^2 - 1)\beta_2 \cos \kappa_2 d_2}{(\beta_2^2 - \beta_1^2)(\kappa_1^2 + \gamma_2^2)} \left[\frac{4\, \gamma_1 \gamma_2}{\beta_1 \beta_2 \left(\frac{n^2 k^2}{\beta_1^2 + n^2 \gamma_1^2} + \gamma_1 d_1 \right) \left(\frac{n^2 k^2}{\beta_2^2 + n^2 \gamma_2^2} + \gamma_2 d_2 \right)} \right]^{\frac{1}{2}}$$

$$\cdot \{ \kappa_1 (k^2 + \beta_1^2 - \beta_2^2) \sin \kappa_1 d_2 - n^2 \gamma_2 k^2 \cos \kappa_1 d_2$$
$$+ n^2 [(\gamma_2 - \gamma_1) k^2 + \gamma_1 (\beta_2^2 - \beta_1^2)] e^{-\gamma_2 (d_1 - d_2)} \cos \kappa_1 d_1 \} \cdot \tag{51}$$

The corresponding expression for the TE modes, equation (37) is apparently considerably simpler.

The expression for the radiation loss of TM modes on a dielectric waveguide of arbitraty shape is obtained from

$$\frac{\Delta P}{P} = \int_{-k}^{k} \{ | q_e(\rho) |^2 + | q_0(\rho) |^2 \} \frac{|\beta|}{\rho} \, d\beta \tag{52}$$

with the coefficient of the even radiation modes

$$q_e(\rho) =$$
$$-\int_0^L \frac{(n^2 - 1)\rho \gamma^{\frac{1}{2}} (\beta_0 \beta \cos \sigma d + \gamma \sigma \sin \sigma d) \cos \kappa d e^{-i(\beta_0 - \beta) z} \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial z} \right)}{2(\beta_0 - \beta) \left\{ \pi \beta_0 |\beta| \left(\frac{n^2 k^2}{\beta_0^2 + n^2 \gamma^2} + \gamma d \right) \left(n^2 \rho^2 \cos^2 \sigma d + \frac{\sigma^2}{n^2} \sin^2 \sigma d \right) \right\}^{\frac{1}{2}}} \, dz \tag{53}$$

and the coefficient for the odd radiation modes

$$q_0(\rho) =$$
$$-\int_0^L \frac{(n^2 - 1)\rho \gamma^{\frac{1}{2}} (\beta_0 \beta \sin \sigma d - \gamma \sigma \cos \sigma d) \cos \kappa d e^{-i(\beta_0 - \beta) z} \left(\frac{\partial f}{\partial z} + \frac{\partial h}{\partial z} \right)}{2(\beta_0 - \beta) \left\{ \pi \beta_0 |\beta| \left(\frac{n^2 k^2}{\beta_0^2 + n^2 \gamma^2} + \gamma d \right) \left(n^2 \rho^2 \sin^2 \sigma d + \frac{\sigma^2}{n^2} \cos^2 \sigma d \right) \right\}^{\frac{1}{2}}} \, dz. \tag{54}$$

The restriction to symmetrical waveguides was dropped so that equations (52) through (54) hold for waveguides of arbitrary sbapes as shown in Fig. 4. A comparison of equations (53) and (45) shows immediately how equation (45) could be generalized to an arbitrary waveguide shape. Corresponding expressions for the odd TE radiation modes could immediately be constructed by a comparison of equation (61), Ref. 1, with equation (54). The function $h(z)$ describes the shape of the dielectric slab waveguide at the lower air-dielectric interface. The theory of dielectric slab waveguides with rough wall, as presented in Ref. 1, was limited to TE modes. The same procedure which lead from equations (45) to (46) allows us to derive the TM-mode radiation loss equations for waveguides with rough walls.

$$q_e(\rho) = \frac{(n^2 - 1)\rho\gamma^{\frac{1}{2}}(\beta_0\beta \cos \sigma d + \gamma\sigma \sin \sigma d) \cos \kappa d[\varphi(\beta) - \psi(\beta)]}{2i\left\{\pi\beta_0 \mid \beta \mid \left(\dfrac{n^2k^2}{\beta_0^2 + n^2\gamma^2} + \gamma d\right)\left(n^2\rho^2 \cos^2 \sigma d + \dfrac{\sigma^2}{n^2} \sin^2 \sigma d\right)\right\}^{\frac{1}{2}}}, \tag{55}$$

and

$$q_0(\rho) = \frac{(n^2 - 1)\rho\gamma^{\frac{1}{2}}(\beta_0\beta \sin \sigma d - \gamma\sigma \cos \sigma d) \cos \kappa d[\varphi(\beta) + \psi(\beta)]}{2i\left\{\pi\beta_0 \mid \beta \mid \left(\dfrac{n^2k^2}{\beta_0^2 + n^2\gamma^2} + \gamma d\right)\left(n^2\rho^2 \sin^2 \sigma d + \dfrac{\sigma^2}{n^2} \cos^2 \sigma d\right)\right\}}. \tag{56}$$

The Fourier component $\varphi(\beta)$ is given by equation (47). The corresponding Fourier component $\psi(\beta)$ follows from equation (47) by replacing $f(z)$ with $h(z)$.

V. NUMERICAL RESULTS

The radiation losses caused by a symmetrical step with the ratio $d_2/d_1 = 0.5$ for $n = 1.01$ are shown in Fig. 5. The solid curves are obtained from equation (42) with the help of equations (45) and (53)

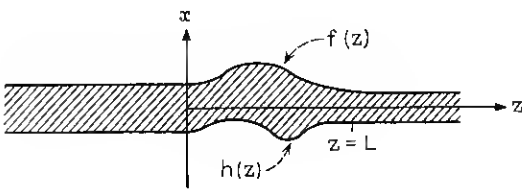

Fig. 4 — An asymmetrical wall distortion of the slab waveguide.

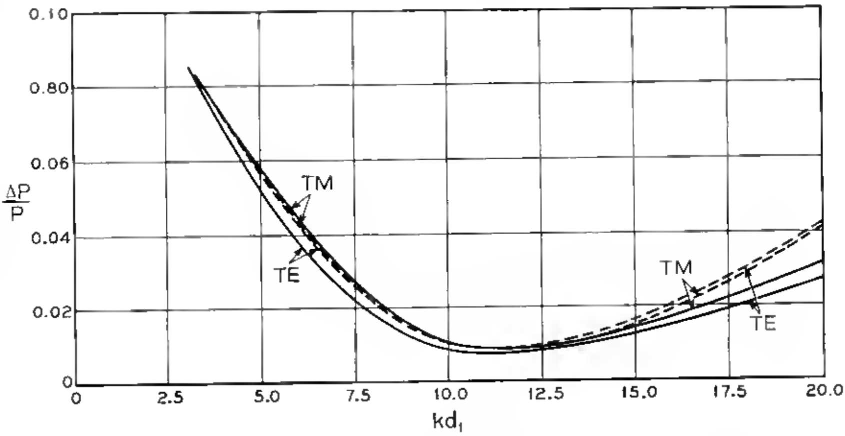

Fig. 5 — TE and TM mode losses caused by a step in the slab waveguide. Solid line calculated from (42), (52) dotted line calculated from (32). $n = 1.01$, $d_2/d_1 = 0.5$.

by approximating the step with a steep linear taper of length $L/d_1 = 1$. For very short tapers, the radiation loss is independent of the length of the taper. The dotted curves were obtained from equations (32) and (37) for TE modes and equations (48) through (51) for TM modes. The agreement between the results obtained by the two different methods is quite good. It is also apparent that TE modes and TM modes suffer very nearly the same losses in this case. It is surprising how low the radiation losses are in the region of $kd_1 = 11$. Both modes pass this considerable step with a power loss of less than 1 percent. For $kd_1 > 20$ the larger portion of the waveguide can support more than one guided mode. This is the reason why the loss curves were not extended past this point. Both the TE as well as TM modes show minimum loss values for particular values of kd_1 suggesting the possibility of optimizing waveguide steps.

Fig. 6 shows the radiation losses of the even, lowest order TE and TM mode for a step on a single mode waveguide with $n = 1.432$. The TE and TM mode losses are quite different for this waveguide with high dielectric constant. The fact that for TE as well as TM modes there is an increasing discrepancy between the two methods of calculation for increasing values of kd, with the dotted curve for the TM modes even becoming negative, may indicate that the solid curves are more reliable. For small values of kd, the agreement between the two methods becomes quite good. The losses of the TM mode are generally higher than the TE mode loss. However, even in this case the TE mode loss can be

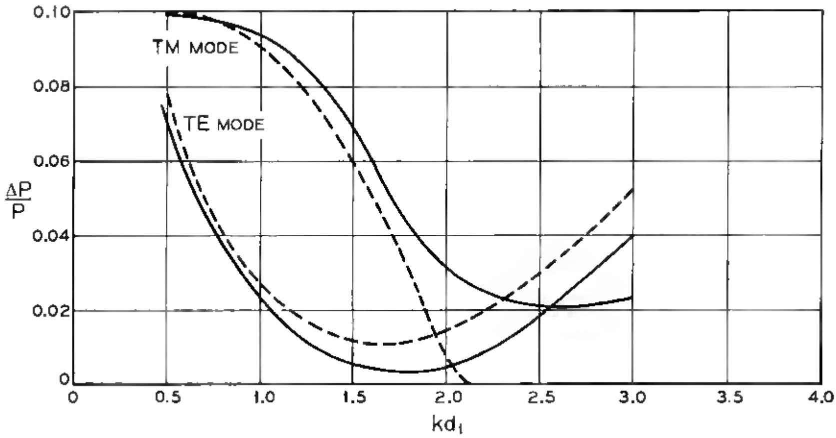

Fig. 6 — Same as Fig. 5. $n = 1.432$.

made approximately 1 percent while the TM mode loss can be as low as 2 percent if the step is used at its optimum point of operation. For $kd_1 > 3$ the larger waveguide section ceases to be single mode.

The dependence of the TE-mode radiation losses on the ratio of the width d_2/d_1 of the guide on either side of the step is shown in Fig. 7. This curve was computed from equations (32), (36) and (37). The dielectric constant of the waveguide material was chosen as $n = 1.01$

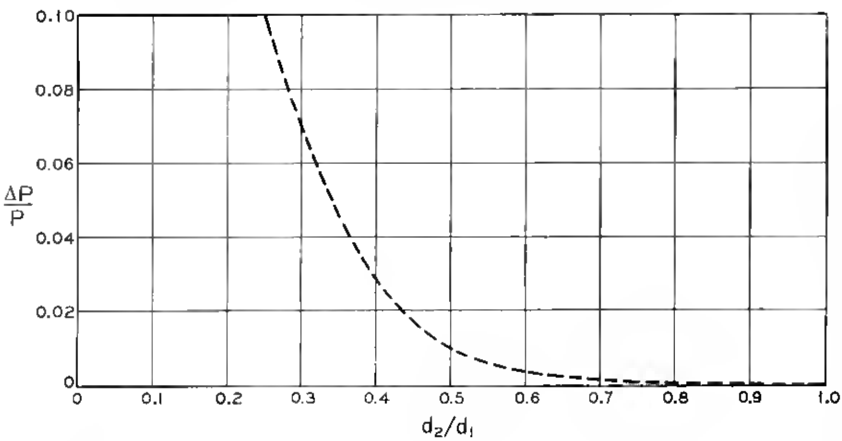

Fig. 7 — Step loss of TE mode as a function of the ratio d_2/d_1. $n = 1.01$, $kd_1 = 10.0$.

and $kd_1 = 10$ was used. It is apparent that the radiation losses increase rapidly as $d_2/d_1 \rightarrow 0$.

So far we have discussed the radiation losses of abrupt steps. The reduction of the TM-mode losses as the step is changed into a taper is seen in Fig. 8. This figure was calculated from equations (52) and (53) for a ratio of $d_2/d_1 = 0.5$ of the straight guide sections that are connected by a symmetrical linear taper. It is apparent that the linear taper needs to be quite long before a substantial improvement of the radiation loss is obtained. The actual length of an effective taper need not be very large. The length of the taper is represented in Fig. 8 as the ratio of its actual length to the half width d_1 of the thicker waveguide section. Extrapolating the result of Fig. 8 to a value of $L/d_1 = 100$ appears to lead to a loss reduction to approximately 1/10 of the loss of the abrupt step. With $\lambda = 1\mu$ we find that $kd_1 = 1$ corresponds to $d_1 = 0.16\ \mu$ so that $L/d_1 = 100$ corresponds to $L = 16\ \mu$.

It appears that there are more effective shapes than linear tapers. Equations (45), (53) and (54) show that the loss of a taper is essentially determined by two factors, the magnitude of the derivatives df/dz and dh/dz and the value of $\beta_0 - \beta$. Rapid oscillations of the function $\exp [i(\beta_0 - \beta)z]$ cause the value of the integral to be small. The largest value of β is $\beta = k$. The worst value appearing in the argument of the exponential function is, therefore, $\beta_0 - k$. The propagation constant of the guided mode depends on the width of the waveguide and is therefore a function of z. The optimum taper, that is intended to connect two

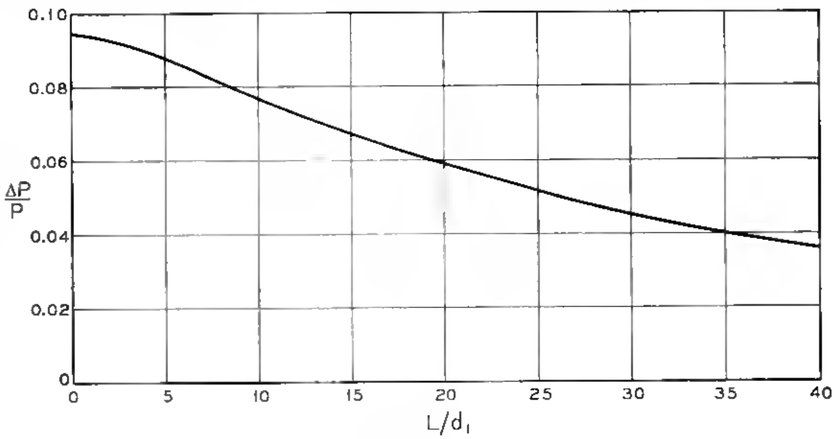

Fig. 8 — TM mode radiation loss as a function of the length L of the taper. $n = 1.432$, $kd_1 = 1.0$, $d_2/d_1 = 0.5$.

different waveguides in a given length, would attempt to use larger values of df/dz and dh/dz on the wide part of the taper where $\beta_0 - k$ is still larger and provide smaller values of these derivatives on its narrow part where $\beta_0 - k$ is smaller. A linear taper radiates more on its narrower portion where the field is less tightly guided. An optimum taper would attempt to distribute the radiation loss uniformly over the length of the taper.

VI. RANDOM WALL DISTORTION

In Ref. 1 we computed the losses of the lowest order guided TE mode that is caused by random distortions of one of the two waveguide walls. For the sake of completeness we include here the corresponding formula for TM modes which can be immediately obtained from the theory presented in Ref. 1 and our present equations (55) and (56). The ensemble average of the relative power loss of the lowest order even TM mode (caused by the distortion of one wall by a random process whose correlation function is a simple exponential function, equation (85) of Ref. 1) with r.m.s. deviation A and correlation length B is given by

$$\left\langle \frac{\Delta P}{P} \right\rangle_{av} = \frac{A^2 \gamma L (n^2 - 1)^2}{2\pi B \beta_0} \int_{-k}^{k} \frac{\rho \cos^2 \kappa_0 d}{\left[(\beta_0 - \beta)^2 + \dfrac{1}{B^2} \right] \left[\dfrac{n^2 k^2}{\beta_0^2 + n^2 \gamma^2} + \gamma d \right]} \cdot \left\{ \frac{(\beta_0 \beta \cos \sigma d + \gamma \sigma \sin \sigma d)^2}{n^2 \rho^2 \cos^2 \sigma d + \dfrac{\sigma^2}{n^2} \sin^2 \sigma d} + \frac{(\beta_0 \beta \sin \sigma d - \gamma \sigma \cos \sigma d)^2}{n^2 \rho^2 \sin^2 \sigma d + \dfrac{\sigma^2}{n^2} \cos^2 \sigma d} \right\} d\beta. \tag{57}$$

The radiation loss that is obtained from this equation is shown in Figs. 9 and 10, by the solid lines. The dotted curves are reproduced from Ref. 1 and give the loss of the TE mode for comparison. The curves labeled $\Delta P^- / \Delta P^+$ show the ratio of backward to forward scattered power. The conclusion to be drawn from these curves is that the TM mode losses caused by small random wall perturbation are very nearly the same as for TE modes. Neither type of mode seems to offer a distinct advantage.

The radiation losses of slab waveguides with random wall distortions are representative of the losses of round waveguides with similar wall distortions. However, the radiation losses of slab waveguide tapers are considerably lower than those of round waveguides. (A discussion of round waveguides will be published.)

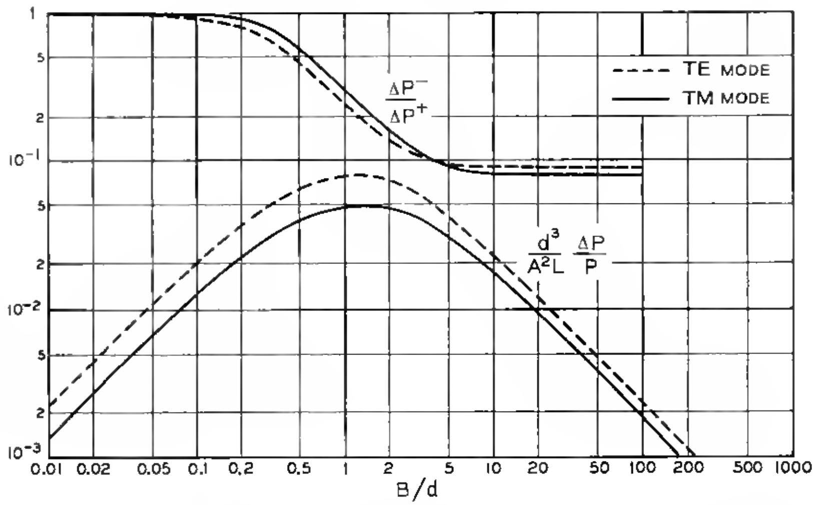

Fig. 9 — Comparison of TM loss (solid line) and TE loss (dotted line) caused by a random distortion of one waveguide wall. B = correlation length, A = r.m.s. wall distortion, d = half width of slab, L = length of distorted guide section. $n = 1.5$, $kd = 1.3$.

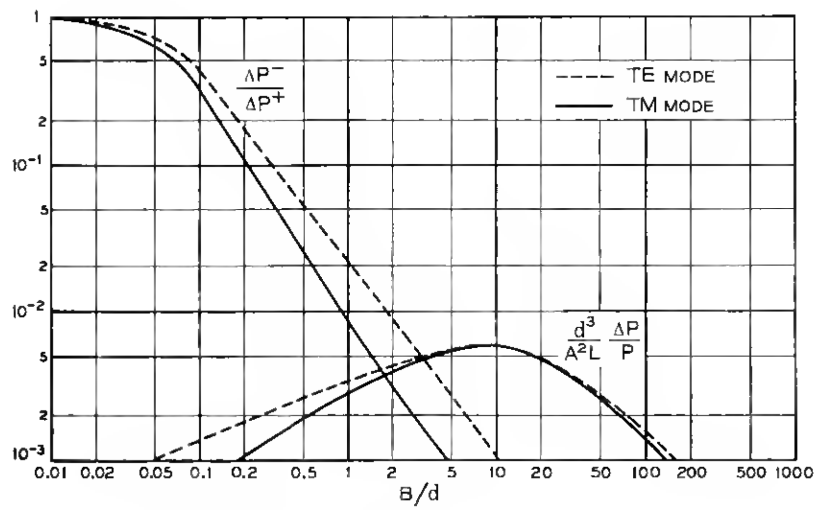

Fig. 10 — Same as Fig. 9. $n = 1.01$, $kd = 8.0$.

VII. CONCLUSION

We have derived radiation loss formulae for the dominant mode dielectric slab waveguide. The losses for steps and tapers in the waveguide were calculated for TE as well as TM modes. The theory of radiation losses for random wall imperfections, that was developed earlier for TE modes, was extended to TM modes.

The radiation losses of abrupt steps with a 2:1 ratio were found to be surprisingly low (a few percent). The advantage of gradual linear tapers over abrupt steps becomes appreciable only if the taper is much longer than the width of the slab.

The losses of steps and tapers of the slab waveguide are exceptionally low. Dielectric waveguides with round and rectangular cross sections have considerably highest losses. However, the method of describing waveguide distortions as successions of abrupt steps is applicable to all dielectric waveguides and simplifies their treatment considerably.

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